Signal Analysis & Communication ECE355

Ch. 1.4 : Unit Impulse & Unit Step functions

Lecture 4

14-09-2023

Ch. 1.4. THE UNIT IMPULSE AND UNIT STEP FUNCTIONS

- Basic building blocks for construction & representation of other signals.
- We will begin with the DT case.

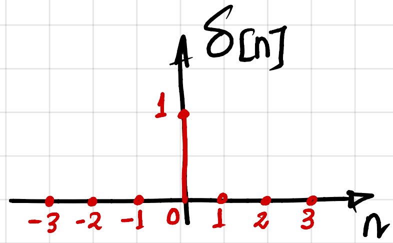
A. DT Unit Impulse & Unit Step Signal/Sequence

a) Unit Impulse (OR Unit Sample)

- Defined as

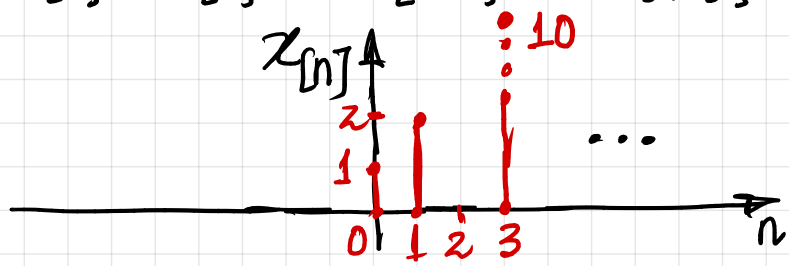
$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

exists when its argument is zero.



- Shifted sig. $\delta[n-k]$ places the impulse at time $k \in \mathbb{Z}$
- useful for building new signals, e.g.,

$$x[n] = \delta[n] + 2\delta[n-1] + 10\delta[n-3] + \dots$$



- Sampling Property of DT Unit Impulse.

$$x[n]\delta[n] = x[0]\delta[n]$$

- Generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] \quad \text{for any } n_0 \in \mathbb{Z}$$

this will be zero at $n=n_0$

- Can be used to sample the value of the sig. at $n=0 \,|\, n=n_0$
- Sifting Property of DT Unit Impulse

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k] \quad \text{for any } k \in \mathbb{Z}$$

- Multiplying any DT sig. by a DT impulse & summing over all time 'picks out' the value of the sig. at the location of the impulse.

- the proof is by direct calculation:

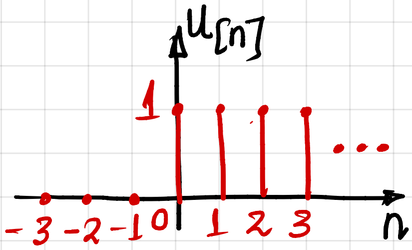
$$\sum_{n=-\infty}^{\infty} \underbrace{\delta[n-k]}_{\text{exists at } n=k} x[n] = \ldots + \underset{\delta[-1]}{0} \cdot x[k-1] + \underset{\delta[0]}{1} \cdot x[k] + \underset{\delta[1]}{0} \cdot x[k+1] + \ldots$$

$$= x[k]$$

b) Unit Step

- Defined as

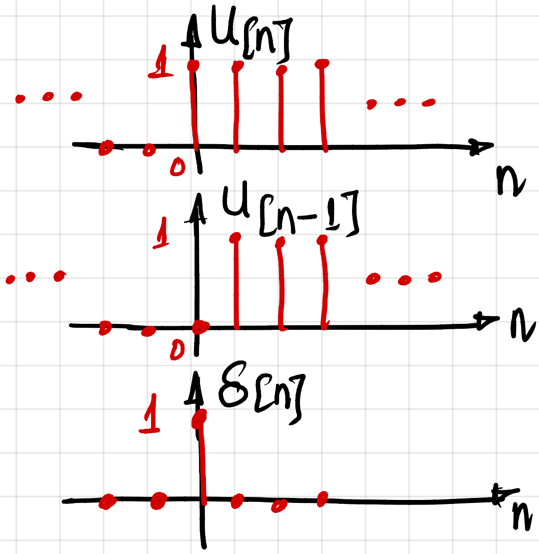
$$u[n] = \begin{cases} 0 & n<0 \\ 1 & n \geq 0 \end{cases}$$



- multiplying by unit step creates a right-sided sig.

Relationship between DT Unit Impulse & Unit Step

- $\delta[n] = u[n] - u[n-1]$



- $$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

for $u[-\infty] \ldots u[-2], u[-1]$:

e.g. $$u[-1] = \sum_{m=-\infty}^{-1} \delta[m] = \ldots + \underset{0}{\delta[-2]} + \underset{0}{\delta[-1]}$$

$$= 0$$

for $u[0]$:

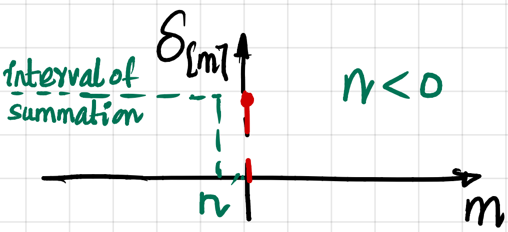
$$u[0] = \sum_{m=-\infty}^{0} \delta[m] = \underset{0}{\ldots} + \underset{0}{\delta[-1]} + \underset{1}{\delta[0]}$$

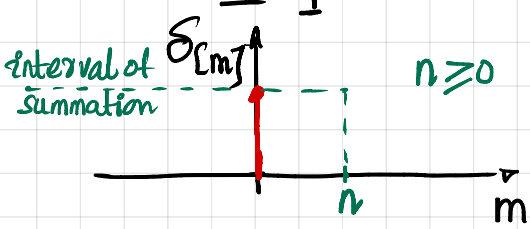
$$= 1$$

for $u[1], u[2], \ldots, u[\infty]$:

$$\text{e.g. } u[1] = \sum_{m=-\infty}^{1} \delta[m] = \underset{0}{\ldots} + \underset{1}{\delta[0]} + \underset{0}{\delta[1]}$$

$$= 1$$





Now assume $k=n-m$

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

$$\delta[n-k] = \begin{cases} 1 & k=n \\ 0 & o/w \end{cases}$$

limits

$m=-\infty$

$k = n-(-\infty)$

$= n+\infty = \infty$

$m=n$

$k = n-n = 0$

Equivalently,

$$\underbrace{u[n] = \sum_{k=0}^{\infty} \delta[n-k]}_{\text{Unit step is the superposition of delayed impulses [we will use later]}}$$

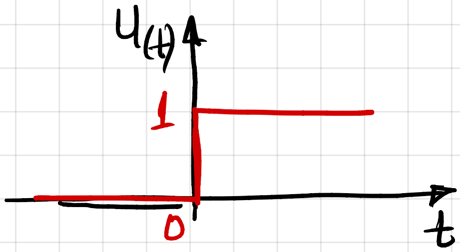
B. CT Unit step & Unit impulse Sig.

- we will define CT unit step first.

ⓐ CT Unit Step

- Defined as

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$



$u(t)$ is discontinuous at $t=0$

- CT unit step is useful for building more complex signals.
- multiplying by the CT unit step creates a right-sided sig.

* for left-sided use $u(-t)$!

(b) CT Unit Impulse

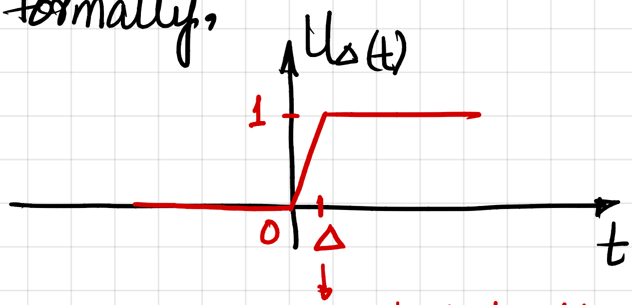
- To define CT unit impulse, let's first look at the analogous relationship b/w $u[n]$ & $\delta[n]$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \quad — (1)$$

- Similarly,

$$\delta(t) = \frac{d}{dt} u(t)$$

- But since $u(t)$ is discontinuous at $t=0$, it is not differentiable.
- More formally,



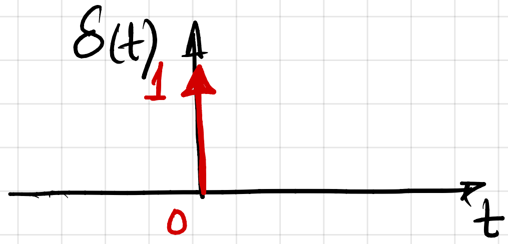
so short duration that it doesn't matter practically.

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

$\Delta \to 0$

short pulse with unit area.

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



- Note:- We can't plot CT unit impulse like a normal sig.
  - It is useful to plot impulse by drawing vertical arrow.
  - $\delta(t)$ is an idealized pulse at $t=0$, which is very fast & very large in size.
- $\delta(t)$ has unit area

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{\Delta \to 0} \int_{-\infty}^{\infty} \delta_\Delta(t)\,dt = 1$$

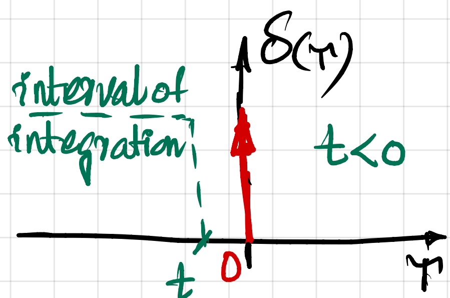
- This is the definition of $\delta(t)$; we understand $\delta(t)$ via how it acts under integral, rather than by the values it takes at any particular $t \in \mathbb{R}$.

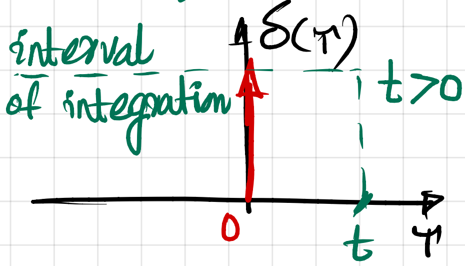
$K\delta(t) \rightarrow$ area K

- As with DT, the simple graphical interpretation of running integral

eqn ①

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$





- With $\sigma = t - \tau$

$$u(t) = \int_{\infty}^{0} \delta(t-\sigma)\,(-d\sigma)$$

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\,d\sigma$$

- Sampling property of CT Unit impulse

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

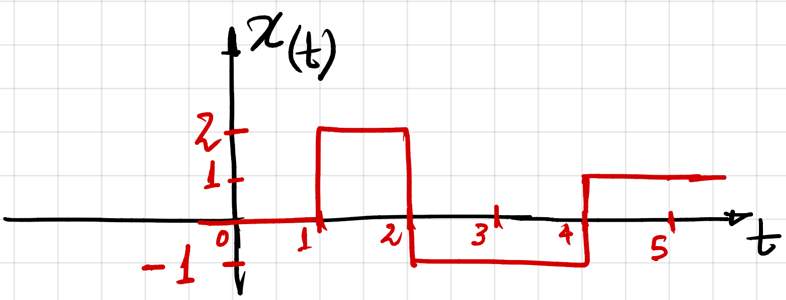
- Sifting property of CT unit impulse

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)\underbrace{\int_{-\infty}^{\infty} \delta(t-t_0)\,dt}_{1}$$

$$= x(t_0)$$

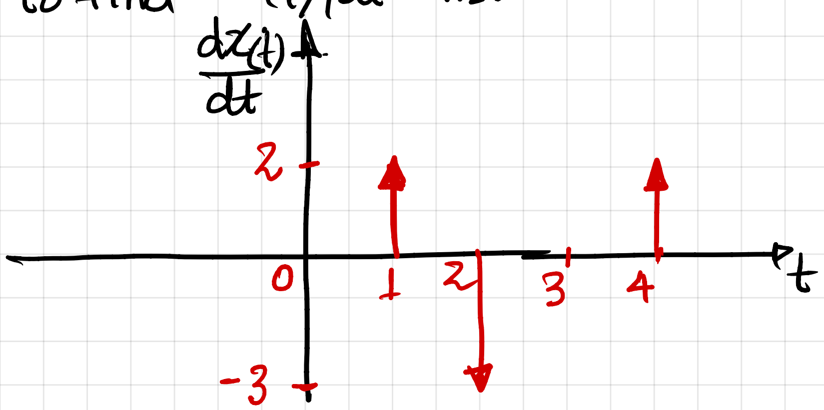
for any $t_0 \in \mathbb{R}$

Example



- To express the sig. mathematically using unit-step.

Step I: Since we know $\delta(t) = \dfrac{du(t)}{dt}$, we may find it convenient to find $dx(t)/dt$ first.



Step II: Express $\frac{dx(t)}{dt}$ mathematically.

$$\frac{dx(t)}{dt} = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$

Step III: Integrate & use relation b/w $\delta(t)$ & $u(t)$:

$$x(t) = \int_{-\infty}^{t} 2\delta(\tau-1)\,d\tau - \int_{-\infty}^{t} 3\delta(\tau-2)\,d\tau + \int_{-\infty}^{t} 2\delta(\tau-4)\,d\tau$$

$$= 2u(t-1) - 3u(t-2) + 2u(t-4)$$

Check: $x(3) = 2 - 3 + 0 = 1$

Application

- CT sig. is converted to DT sig. by Sampling (covered later)
- Sampling makes use of impulse train sig., given as:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$